Math 2010

§ Taylor's Formula for two variables:

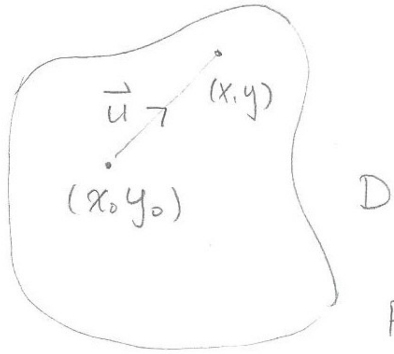
Let $f: D \to \mathbb{R}$ be a smooth function (analytic).

$(x_0, y_0), (x, y) \in D$ (See Pic. 1)

We define $\vec{v} := (x - x_0, y - y_0) = (v_1, v_2)$, $\vec{u} = \dfrac{\vec{v}}{|\vec{v}|}$.

$r(t) := (x_0, y_0) + t\vec{u}$

$g(t) := f(r(t))$



Pic. 1

So $\begin{cases} g(0) = f(x_0, y_0) \\ g(|\vec{v}|) = f(x, y) \end{cases}$

Suppose that $g(t) = \displaystyle\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} \cdot t^n$ Taylor expansion for 1-variable function.

then $f(x, y) = g(|\vec{v}|) = \displaystyle\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} \cdot |\vec{v}|^n$

Now, $g^{(n)}(0) = D_u^n f(x_0, y_0)$

$$= \left(u_1 \frac{\partial}{\partial x} + u_2 \frac{\partial}{\partial y} \right)^n f \Big|_{(x_0, y_0)}$$

So $g^{(n)}(0) \cdot |\vec{v}|^n = \left(|\vec{v}| u_1 \dfrac{\partial}{\partial x} + |\vec{v}| u_2 \dfrac{\partial}{\partial y} \right)^n f \Big|_{(x_0, y_0)}$

$$= \left(v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} \right)^n f \Big|_{(x_0, y_0)}$$

$$\left(V_1 \frac{\partial}{\partial x} + V_2 \frac{\partial}{\partial y} \right)^n$$

$$= V_1^n \partial_x^n + \binom{n}{1} V_1^{n-1} V_2 \cdot \partial_x^{n-1} \partial_y + \binom{n}{2} V_1^{n-2} V_2^2 \partial_x^{n-2} \partial_y^2$$

$$+ \cdots + V_2^n \partial_y^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} \cdot V_1^{n-k} \cdot V_2^{k} \cdot \partial_x^{n-k} \partial_y^{k}$$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}.$$

Now, $V_1 = (x - x_0)$, $V = (y - y_0)$, so

$$f(x,y) = f(x_0 y_0) + \partial_x f(x_0 y_0)(x - x_0) + \partial_y f(x_0 y_0)(y - y_0)$$

$$+ \frac{1}{2!} \Big(\partial_x^2 f(x_0 y_0)(x - x_0)^2 + 2 \partial_x \partial_y f(x_0 y_0)(x - x_0)(y - y_0)$$

$$+ \partial_y^2 f(x_0 y_0)(y - y_0)^2 \Big)$$

$$+ \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \sum_{k=0}^{n} \binom{n}{k} \cdot \partial_x^{n-k} \partial_y^{k} f(x_0 y_0)(x - x_0)^{n-k} (y - y_0)^{k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!\,k!} \partial_x^{n-k} \partial_y^{k} f(x_0 y_0)(x - x_0)^{n-k} (y - y_0)^{k}.$$

Mean Value Theorem and Error Bounds.

For one variable functions, mean value theorem tells us

$$\frac{f(x)-f(y)}{(x-y)} = f'(\xi)$$

for some $\xi \in (x, y)$.

So $|f(y) - f(x)| \leq |f'(\xi)| \cdot |x-y|$

$$\leq \max_{t\in(x,y)} |f'(t)|\,|x-y|.$$

Inductively, we can find the error bound for the Taylor expansion: (We need FTC, too)

Let $R_{n+1}(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$

So $\left(\frac{d}{dx}\right)^p R_{n+1}(x_0) = 0$ for $p = 1, 2 \ldots n$

$$\left|\int_{x_0}^{x} \left(\frac{d}{dx}\right)^n R_{n+1}(x) - \left(\frac{d}{dx}\right)^n R_{n+1}(x_0) \, dx\right| = \left|\left(\frac{d}{dx}\right)^{n-1} R_{n+1}(x)\right|$$

$$= \left|\int_{x_0}^{x} \left(\frac{d}{dx}\right)^{n+1} f(\xi) \cdot (x-x_0) \, dx\right| \leq \frac{1}{2} \max_{t\in(x_0,x)} |f^{(n+1)}(t)| \cdot |x-x_0|^2$$

Integrate again :

$$\left|\left(\frac{d}{dx}\right)^{n-2} R_{n+1}(x)\right| \leq \int_{x_0}^{x} \frac{1}{2} \max_{t\in(x_0,x)} |f^{(n+1)}(t)| (x-x_0)^2 \, dx$$

$$\leq \frac{1}{3!} \max_{t\in(x_0,x)} |f^{(n+1)}(t)| \cdot |x-x_0|^3$$

and so on , we finally have

$$|R_{n+1}(x)| \leq \frac{1}{(n+1)!} \max_{t\in(x_0,x)} |f^{(n+1)}(t)| \, |x-x_0|^{n+1}$$

Now, for the function of two variables :

$$|R_{n+1}(x,y)| \leq \frac{1}{(n+1)!} \cdot \max_{t\in(0,|\vec{v}|)} |D_{\vec{u}}^{(n+1)} f(r(t))| \, |\vec{v}|^{n+1}$$

$\max\limits_{\substack{t\in(0,|\vec{v}|) \\ |\vec{u}|=1}} D_{\vec{u}}^{(n+1)} f(r(t))$ is very hard to evalue.

When $n=0$,

$$\max_{t\in(0,|\vec{v}|)} |D_{\vec{u}} (f(r(t)))| = \max |\nabla f \cdot \vec{u}| \leq \max\left(|\frac{\partial}{\partial x} f| + |\frac{\partial}{\partial y} f|\right)$$

(we can take maximum in the domain of f).

For $n=1$, we have

$$\max |D_{\vec{u}}^2 f| = \max | \partial_x^2 f \, u_1^2 + 2\, \partial_x \partial_y f \, u_1 u_2 + \partial_y^2 f \, u_2^2 |$$

So $\max |D_{\vec{u}}^2 f| \cdot |\vec{v}|^2$

$$\leq \max | \partial_x^2 f \, |x-x_0|^2 + 2\, \partial_x \partial_y f \, |x-x_0||y-y_0| + \partial_y^2 f \, |y-y_0|^2 |$$

So

$$|R_2(x)| \leq \frac{1}{2} \max(|\partial_x^2 f|, |\partial_x \partial_y f|, |\partial_y^2 f|) \cdot (|x-x_0| + |y-y_0|)^2$$

For general n, we have

$$|R_{n+1}(x)| \leq \frac{1}{(n+1)!} \cdot \max(|\partial_x^n f|, |\partial_x^{n-1} \partial_y f|, \ldots, |\partial_y^n f|) \cdot (|x-x_0| + |y-y_0|)^{n+1}$$